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THE POTENTIAL OF RING-SHAPED DISCS.

By E. P. Adams.

The method of obtaining solutions of Laplace's equation in cases where there is symmetry about an axis by means of zonal harmonics is well known. In brief, the method consists in expanding the known value of the potential on the axis in a power series in x, the distance measured along the axis, and in this series replacing x^n and $1/x^{n+1}$ by $r^n P_n (\cos \theta)$ and $(1/r^{n+1}) P_n (\cos \theta)$ respectively. r is the distance in any direction, θ, measured from the origin. Each term in the result being a solution of Laplace's equation which reduces to the given value on the axis, it follows that the expression so obtained is the value of the potential at any point where the series is convergent. There are cases in which this method fails. We can, however, express the potential at any point due to an elementary ring of our distribution of matter by this method, and then in a number of important cases this expression can be integrated over the whole distribution in such a way that the resulting series is absolutely convergent for a definite region. In cases where both methods can be used some interesting consequences follow from a comparison of the results.

In the following we shall be concerned with thin circular ring-shaped discs of both single and double layers; we shall assume that the density in the former case and the strength in the latter case is a function of the distance from the center of the disc only. The internal and external radii of the disc are a and b respectively; ρ is the variable radius measured in the disc; and r, θ, the polar coördinates of any point in space measured with respect to the x axis which is supposed drawn normal to the plane of the disc from its center. As we deal only with cases of axial symmetry the second angular coördinate of any point in space is not required.

I. Potential due to single layer distributions.

1. Denoting the density of the distribution by $f(\rho)$ we have for the potential at any point on the axis:

$$V = 2\pi \int_a^b \frac{f(\rho) \rho d\rho}{\sqrt{x^2 + \rho^2}}. \tag{1}$$

We have to consider three regions:

1. $0 \leqq r \leqq a$,
2. $a \leqq r \leqq b$,
3. $b \leqq r$.

By first expressing the potential due to an elementary ring at any point within these three regions by the proper expansion in terms of zonal harmonics, and then integrating over the disc, we get:

$0 \leqq r \leqq a$

$$V = 2\pi \int_a^b f(\rho) \left\{ 1 - \frac{1}{2}\frac{r^2}{\rho^2}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{\rho^4}P_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{r^6}{\rho^6}P_6 + \cdots \right\} d\rho, \tag{2}$$

$r \geqq b$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{\rho}{r} - \frac{1}{2}\frac{\rho^3}{r^3}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{\rho^5}{r^5}P_4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{\rho^7}{r^7}P_6 + \cdots \right\} d\rho, \tag{3}$$

$a \leqq r \leqq b$

$$V = 2\pi \int_a^r f(\rho) \left\{ \frac{\rho}{r} - \frac{1}{2}\frac{\rho^3}{r^3}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{\rho^5}{r^5}P_4 - \cdots \right\} d\rho$$

$$+ 2\pi \int_r^b f(\rho) \left\{ 1 - \frac{1}{2}\frac{r^2}{\rho^2}P_2 + \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{\rho^4}P_4 - \cdots \right\} d\rho. \tag{4}$$

In these expressions P_n is written for $P_n(\cos\theta)$.

In any case, therefore, in which we can integrate $\int \rho^n f(\rho) d\rho$, where n is any positive or negative integer we can solve the problem completely.

Now we know that the resulting expression for the potential in any one of the three regions must be a solution of Laplace's equation:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0. \tag{5}$$

Terms of the form $A_n r^n P_n$ and $(B_n/r^{n+1})P_n$ are known to be solutions of this equation. Therefore the sum of all the other terms must be a solution; in this way we are able to get some interesting special solutions. Our results must be continuous at the boundaries of the separate regions. The solution in the region $a \leqq r \leqq b$ must satisfy the relation at $x = 0$, $\theta = \pi/2$:

$$-\frac{\partial V}{\partial x} = 2\pi f(\rho). \tag{6}$$

Potential due to single layer distributions whose density is proportional to a positive or negative integral power of the radius.

2. Before giving the solution of this case in general it will be worth while to consider two special cases. First, suppose the disc to be of constant density, k. Applying (2), (3), and (4) we find:

$0 \leqq r \leqq a$

$$V = 2\pi k \left\{ (b - a) + \frac{1}{2}\left(\frac{1}{b} - \frac{1}{a}\right) r^2 P_2 - \frac{1 \cdot 1}{2 \cdot 4}\left(\frac{1}{b^3} - \frac{1}{a^3}\right) r^4 P_4 + \cdots \right\}, \tag{7}$$

$r \geqq b$

$$V = 2\pi k \left\{ \frac{1}{2}\frac{b^2 - a^2}{r} - \frac{1 \cdot 1}{2 \cdot 4}\frac{b^4 - a^4}{r^3} P_2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{b^6 - a^6}{r^5} P_4 - \cdots \right\}, \tag{8}$$

$a \leqq r \leqq b$

$$\begin{aligned} V = 2\pi k \Big\{ & b\left(1 + \frac{1}{2}\frac{r^2}{b^2} P_2 - \frac{1 \cdot 1}{2 \cdot 4}\frac{r^4}{b^4} P_4 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{r^6}{b^6} P_6 - \cdots\right) \\ & - a\left(\frac{1}{2}\frac{a}{r} - \frac{1 \cdot 1}{2 \cdot 4}\frac{a^3}{r^3} P_2 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\frac{a^5}{r^5} P_4 - \cdots\right) \\ & - r\left(\frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{4}\right) P_2 - \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{3} + \frac{1}{6}\right) P_4 \right. \\ & \left. + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{5} + \frac{1}{8}\right) P_6 - \cdots\right) \Big\}. \end{aligned} \tag{9}$$

Putting $r = a$ in (7) and (9) we see that they both give the same result; and putting $r = b$ in (8) and (9) we get the same result. Thus the conditions of continuity are satisfied. The last term in (9) must be a solution of Laplace's equation; we should therefore expect this term to be $rP_1(\cos\theta)$. This is the value we should get for it if we had solved this problem by integrating (1) first and then expanding it in terms of zonal harmonics in the usual way. Further, with this value the relation (6) is satisfied. We therefore have:

$$\begin{aligned} P_1 = \cos\theta = \frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{4}\right) P_2 - \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{3} + \frac{1}{6}\right) P_4 \\ + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\left(\frac{1}{5} + \frac{1}{8}\right) P_6 - \cdots. \end{aligned} \tag{10}$$

The infinite series in (10) is absolutely convergent in the interval $-\pi/2 \leqq \theta \leqq +\pi/2$.

3. We next consider the case where the density is inversely proportional to the distance from the center of the disc. According to the usual method we should have to develop (1) in a series, with $f(\rho) = k\rho^{-1}$, after integrating. On the axis we find:

$$V = 2\pi k \log \frac{b + \sqrt{b^2 + x^2}}{a + \sqrt{a^2 + x^2}}.$$

There is no difficulty here provided that $x > b$ or $x < a$. But in the region $b > x > a$, we find:

$$V = 2\pi k\left\{\log\frac{2b}{x} + \frac{1\cdot 1}{2\cdot 2}\frac{x^2}{b^2} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{x^4}{b^4} + \cdots \right.$$
$$\left. - \frac{a}{x} + \frac{1\cdot 1}{2\cdot 3}\frac{a^3}{x^3} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^5}{x^5} + \cdots\right\}. \quad (11)$$

Here it is the term $\log 2b/x$ that causes difficulty. It might be thought that we could use the zonal harmonics of the second kind, but these become infinite on the axis and so are useless for these problems. If we use (4) we get for the potential at any point in the region $a < r < b$:

$$V = 2\pi k\left\{\log\frac{b}{r} + 1 - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{5}\right)P_4 - \cdots \right.$$
$$+ \frac{1\cdot 1}{2\cdot 2}\frac{r^2}{b^2}P_2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{r^4}{b^4}P_4 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\frac{r^6}{b^6}P_6 - \cdots \quad (12)$$
$$\left. - \frac{a}{r} + \frac{1\cdot 1}{2\cdot 3}\frac{a^3}{r^3}P_2 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^5}{r^5}P_4 + \cdots\right\}.$$

Comparing (11) and (12) we see that $\log 2b/x$ on the axis expands into $\log(b/r) + L_0(\cos\theta)$, where

$$L_0 = 1 - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{5}\right)P_4 - \cdots. \quad (13)$$

The other terms in (12) are formed from the corresponding terms in (11) in the usual way. Now from the fact that $L_0 - \log r$ must be a solution of (5), we find that L_0 satisfies the equation:

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial L}{\partial\theta}\right) = \sin\theta$$

whose complete solution is:

$$L_0 = c_1\log\tan\frac{\theta}{2} - \log\sin\theta + c_2.$$

Since L_0 must remain finite on the axis we must have $c_1 = 1$. Further, we see that we must have:

$$\log 2 = 1 - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{5}\right) - \cdots \quad (14)$$

and that:

$$L_0 = \log\frac{4}{1+\cos\theta} \quad (15)$$

$\log r(1+\cos\theta)$ is thus a solution of Laplace's equation, which is easily verified. It may be shown without difficulty that the conditions of continuity are satisfied, as well as the relation (6).

4. In considering the general case where the density varies as any integral power of the radius it is necessary to take separately even and odd, and positive and negative powers. Let us consider first the case where the density is equal to $k\rho^{2n}$. On the axis we have:

$$V = 2\pi k \int_a^b \frac{\rho^{2n+1} d\rho}{\sqrt{x^2 + \rho^2}}.$$

This can be integrated, giving:

$$V = \frac{2\pi k}{2n+1}\left[\sqrt{x^2+\rho^2}\left(\rho^{2n} - \frac{2nx^2\rho^{2n-2}}{(2n-1)} + \frac{2n(2n-2)x^4\rho^{2n-4}}{(2n-1)(2n-3)} - \cdots \right.\right.$$
$$\left.\left. + (-1)^n \frac{2n(2n-2)(2n-4)\cdots 2}{(2n-1)(2n-3)\cdots 1} x^{2n}\right)\right]_a^b$$

In this expression, after putting in the limits, $\sqrt{x^2 + b^2}$ is to be developed by the binomial theorem in an infinite series in x/b, and $\sqrt{x^2 + a^2}$ in an infinite series in a/x. After multiplying the finite sum by the two infinite series, x^n is to be replaced by $r^n P_n(\cos\theta)$ and x^{-n} by $(1/r^n)P_{n-1}(\cos\theta)$ and the result will be the solution of the problem in the region $a \leqq r \leqq b$. This problem can also be solved by applying (4) to this case, putting $f(\rho) = k\rho^{2n}$. On comparing the results of these two solutions we find that we must have:

$$\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n+1)} P_{2n+1}$$
$$= \sum_{m=0}^{\infty} (-1)^{m+n+1} \frac{1\cdot 3\cdot 5\cdots 2m-1}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n-1} + \frac{1}{2m+2n+2}\right) P_{2m}. \tag{16}$$

We are thus able to express any zonal harmonic of odd integral order as an absolutely convergent infinite series, valid between $\theta = -\pi/2$ and $\theta = +\pi/2$, in all the zonal harmonics of even integral order. Equation (10) is seen to be a special case of this general formula for $n = 0$. We get the same result by taking the density to be equal to $k\rho^{-2n-3}$.

5. When we consider simple layers whose density is inversely proportional to an even power or directly proportional to an odd power of the distance from the center we get results of an entirely different character. Assuming the density to be equal to k/ρ^{2n+2}, we find by the methods already employed:

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log 2 = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\left(\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\cdots\right.$$
$$\left.+\frac{1}{(2n-1)2n}\right)+\sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n}\right. \tag{17}$$
$$\left.+\frac{1}{2m+2n+1}\right).$$

In this expression n may be any positive integer including 0. It should be noted that the first term in the parenthesis in the infinite series becomes infinite for the particular value of $m = n$. This term is to be ignored for that value of m. This same result is obtained when the density is taken to be equal to $k\rho^{2n-1}$. With $n = 0$ we get (14) as a special case.

With the density proportional to ρ^{2n+2}, this method leads to a particular solution of Laplace's equation:

$$V = \frac{1}{r^{2n+1}}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log r\cdot P_{2n} + L_{2n}\right). \tag{18}$$

Taking the density proportional to ρ^{2n-1} we find the corresponding solution of Laplace's equation:

$$V = r^{2n}\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\log r\cdot P_{2n} - L_{2n}\right). \tag{19}$$

In these expressions L_{2n} is the absolutely convergent series:

$$L_{2n} = \sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot 3\cdot 5\cdots(2m-1)}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n}+\frac{1}{2m+2n+1}\right)P_{2m}. \tag{20}$$

By substituting either (18) or (19) in Laplace's equation (5) we find that L_{2n} satisfies the equation:

$$\frac{\partial^2 y}{\partial\theta^2}+\cot\theta\frac{\partial y}{\partial\theta}+2n(2n+1)y = (4n+1)\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}P_{2n}. \tag{21}$$

6. As we shall meet similar equations in some of the other applications of this method let us write it, putting $\cos\theta = \mu$, and $2n = m$,

$$(1-\mu^2)\frac{\partial^2 y}{\partial\mu^2}-2\mu\frac{\partial y}{\partial\mu}+m(m+1)y = AP_m(\mu), \tag{22}$$

where A is a numerical constant. This equation with the right-hand member equal to 0 is Legendre's equation whose complete solution we know. Equation (22) can therefore be solved by the method of variation of parameters; we find for its solution:

$$y = c_1Q_m + c_2P_m + AQ_m\int P_m^2 d\mu - AP_m\int P_mQ_m d\mu,$$

where Q_m is the zonal harmonic of order m of the second kind. But as we are concerned only with solutions which remain finite on the axis one of the integration constants must be determined so as to satisfy this condition. This leads to a solution of the form:

$$y = a_mP_m \cdot \log(1 + \mu) + G + c_2P_m,$$

where G is a polynomial of degree $m - 1$. Since any polynomial can be expressed as a finite sum of zonal harmonics of decreasing order beginning with that one of order equal to the degree of the polynomial, we can get our solution quickly by assuming:

$$y = a_mP_m \log(1 + \mu) + c_2P_m + \sum_{k=1}^{m} a_{m-k}P_{m-k}. \tag{23}$$

When this is substituted in (22) we find:

$$\begin{aligned} a_m &= -\frac{A}{2m+1}, \\ a_{m-k} &= -(-1)^k \frac{2A}{2m+1}\,\frac{2m-2k+1}{k(2m-k+1)}. \end{aligned} \tag{24}$$

The one remaining constant, c_2, may be found from the condition for $\theta = 0$. The value of this constant for the solution of (21) we find from (17), (20) and (23):

$$c_2 = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\,2\log 2 - \sum_{k=1}^{2n} a_{2n-k} \\ - \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots2n}\left(\frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \cdots + \frac{1}{(2n-1)2n}\right).$$

7. Although the infinite series represented by (16) and (20) are absolutely convergent for $-\pi/2 \leq \theta \leq +\pi/2$, their derivatives are not convergent. Now it is the infinite series that we get when we apply (4) to determine the value of the potential at any point in the region $a \leq r \leq b$. In order that (6) may be satisfied it is necessary to substitute for the infinite series their equivalents given by (21) and (24). Having done this all the conditions of the problem will be satisfied.

Potential due to single layer distributions whose density is a logarithmic function of the radius.

8. If we take the density of the single layer distribution to be proportional to $\log \rho/\rho^{2n+3}$, where n is any positive integer including 0, it is possible to integrate the potential on the axis and then expand it in a

series which will be convergent in the region $a \leq r \leq b$. We can also apply (4) to this case directly. On comparing the results obtained by the two methods we find the following:

If we define the infinite series:

$$M_{2n+1} = \sum_{m=0}^{\infty} (-1)^{n+m} \frac{1\cdot3\cdot5\cdots(2m-1)}{2\cdot4\cdot6\cdots2m} \left\{ \frac{1}{(2n-2m+1)^2} - \frac{1}{(2n+2m+2)^2} \right\} P_{2m} \qquad (25)$$

we have a solution of Laplace's equation:

$$V = \frac{1}{r^{2n+2}} \left\{ \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \log r \cdot P_{2n+1} + M_{2n+1} \right\} \qquad (26)$$

and M_{2n+1} satisfies the equation:

$$\frac{\partial^2 y}{\partial \theta^2} + \cot\theta \frac{\partial y}{\partial \theta} + (2n+2)(2n+1)y = (4n+3)\frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} P_{2n+1}.$$

Putting $\cos\theta = \mu$, and $2n + 1 = m$, we get equation (22) whose solution we have found subject to the condition of remaining finite on the axis. We find in addition that the following relation must hold:

$$\frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \log 2 = \frac{2\cdot4\cdot6\cdots2n}{1\cdot3\cdot5\cdots(2n+1)} \left\{ \frac{1}{2n+1} + \frac{1}{2n(2n-1)} + \frac{1}{(2n-2)(2n-3)} + \cdots + \frac{1}{2\cdot1} \right\}$$
$$- \sum_{m=0}^{\infty} (-1)^{n+m} \frac{1\cdot3\cdot5\cdots2m-1}{2\cdot4\cdot6\cdots2m} \left\{ \frac{1}{(2n-2m+1)^2} - \frac{1}{(2n+2m+2)^2} \right\}.$$

Particular cases of this last formula for $n = 0$ and $n = 1$ are:

$$\log 2 = 1 - \left(1 - \frac{1}{2^2}\right) + \frac{1}{2}\left(1 - \frac{1}{4^2}\right) - \frac{1\cdot3}{2\cdot4}\left(\frac{1}{3^2} - \frac{1}{6^2}\right) + \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{5^2} - \frac{1}{8^2}\right) - \cdots,$$

$$\frac{2}{3}\log 2 = \frac{5}{9} + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) - \frac{1}{2}\left(1 - \frac{1}{6^2}\right) + \frac{1\cdot3}{2\cdot4}\left(1 - \frac{1}{8^2}\right) - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{3^2} - \frac{1}{10^2}\right) + \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\left(\frac{1}{5^2} - \frac{1}{12^2}\right) - \cdots.$$

9. If we take the density of the single layer distribution to be proportional to $\rho^{2n} \log \rho$, where n is any positive integer, including 0, we

get the same results with the exception of the particular solution of Laplace's equation. Corresponding to (26) we now find a solution:

$$V = \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)}\, r^{2n+1}\log r\cdot P_{2n+1} - r^{2n+1}M_{2n+1},$$

where M_{2n+1} is the infinite series (25).

10. In all the problems considered so far it has been possible to integrate the value of the potential on the axis in terms of the elementary functions, and our results have been obtained by comparing the expansion of this expression with the one arising from the application of equation (4). But there are many cases in which this cannot be done, and it is of some interest to see what results we get from the use of (4) alone.

Assume the density of the single layer to be $k(\log \rho/\rho)$. Using (4) we find for the potential in the region $a \leqq r \leqq b$:

$$\begin{aligned} V = 2\pi k \Big\{ & \log r\left[1 - \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)P_2 + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{5}\right)P_4 - \cdots\right] \\ & - \left[1 + \frac{1}{2}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4^2}-\frac{1}{5^2}\right)P_4 + \cdots\right] \\ & - \frac{1}{2}(\log r)^2 + \frac{1}{2}(\log b)^2 \\ & - \frac{a}{r}(\log a - 1) + \frac{1}{2}\frac{a^3}{r^3}\left(\frac{\log a}{3} - \frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\frac{a^5}{r^5}\left(\frac{\log a}{5} - \frac{1}{5^2}\right)P_4 + \cdots \\ & + \frac{1}{2}\frac{r^2}{b^2}\left(\frac{\log b}{2} + \frac{1}{2^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\frac{r^4}{b^4}\left(\frac{\log b}{4} + \frac{1}{4^2}\right)P_4 + \cdots \Big\}. \end{aligned}$$

The coefficient of $\log r$ we have already had; it is the L_0 of (13) whose value we have found to be:

$$L_0 = \log \frac{4}{1+\cos\theta}.$$

Using this value we find that (6) is satisfied by this solution. It is now of interest to try to determine the value of the other infinite series of zonal harmonics which enters into this solution. Writing this:

$$f_0 = 1 + \frac{1}{2}\left(\frac{1}{2^2}-\frac{1}{3^2}\right)P_2 - \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{4^2}-\frac{1}{5^2}\right)P_4 + \cdots \tag{27}$$

we see that:

$$V = \log r\cdot\log\frac{4}{1+\cos\theta} - f_0 - \frac{1}{2}(\log r)^2$$

is a solution of Laplace's equation since all the other terms are separately solutions. We thus find that f_0 satisfies the equation

$$\frac{\partial^2 f_0}{\partial\theta^2} + \cot\theta\frac{\partial f_0}{\partial\theta} + 1 = \log\frac{4}{1+\cos\theta}.$$

This equation cannot be solved in terms of the elementary functions. By the usual methods we find a solution with one arbitrary constant determined so that f_0 will remain finite on the axis:

$$f_0 = c_2 - \log 2 \log(1+\cos\theta) - \left(\frac{1-\cos\theta}{2}\right) - \frac{1}{2^2}\left(\frac{1-\cos\theta}{2}\right)^2 - \frac{1}{3^2}\left(\frac{1-\cos\theta}{2}\right)^3 - \cdots.$$

When $\theta = 0$,

$$f_0 = c_2 - (\log 2)^2.$$

From (27) for $\theta = 0$, we find, accurate to four places:

$$f_0 = 1.0627\cdots.$$

Hence

$$c_2 = 1.5432\cdots.$$

For $\theta = \pi/2$, we have:

$$f_0 = c_2 - \frac{1}{2} - \frac{1}{2^2 2^2} - \frac{1}{3^2 2^3} - \frac{1}{4^2 2^4} - \cdots.$$

Using the known series:

$$\sum_{m=1}^{\infty}\frac{1}{2^m m^2} = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

we find:

$$0.9609\cdots = 1 - \left(\frac{1}{2}\right)^2\left(\frac{1}{2^2} - \frac{1}{3^2}\right) - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{1}{4^2} - \frac{1}{5^2}\right) - \cdots.$$

II. Potential due to double layer distributions.

11. The potential at any point due to a double layer distribution of unit strength is equal to the solid angle subtended by the distribution at the point. We shall again consider a circular ring-shaped disc of internal and external radii a and b, whose strength, $f(\rho)$ is a function of the distance from the center only.

The solid angle subtended by an elementary ring of radius ρ and width $d\rho$ at a point, x, on the axis is

$$d\omega = 2\pi\frac{x\rho d\rho}{(x^2+\rho^2)^{3/2}}.$$

We accordingly get for the potential at any point on the axis:

$$V = 2\pi x\int_a^b f(\rho)\frac{\rho d\rho}{(x^2+\rho^2)^{3/2}}. \tag{28}$$

By developing the potential due to an elementary ring of the distribution in a series of zonal harmonics, and then integrating over the disc, we get in the three regions:

$r < a$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{r}{\rho^2} P_1 - \frac{3}{2} \frac{r^3}{\rho^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{r^5}{\rho^6} P_5 - \cdots \right\} d\rho, \tag{29}$$

$r > b$

$$V = 2\pi \int_a^b f(\rho) \left\{ \frac{\rho}{r^2} P_1 - \frac{3}{2} \frac{\rho^3}{r^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{\rho^5}{r^6} P_5 - \cdots \right\} d\rho, \tag{30}$$

$a < r < b$

$$\begin{aligned} V = 2\pi \int_a^r f(\rho) &\left\{ \frac{\rho}{r^2} P_1 - \frac{3}{2} \frac{\rho^3}{r^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{\rho^5}{r^6} P_5 - \cdots \right\} d\rho \\ &+ 2\pi \int_r^b f(\rho) \left\{ \frac{r}{\rho^2} P_1 - \frac{3}{2} \frac{r^3}{\rho^4} P_3 + \frac{3 \cdot 5}{2 \cdot 4} \frac{r^5}{\rho^6} P_5 - \cdots \right\} d\rho. \end{aligned} \tag{31}$$

As in the problem of single layer distributions V must be a solution of Laplace's equation (5); there must be continuity at $r = a$ between the results of (29) and (31), and between (30) and (31) at $r = b$. In place of relation (6) we must now have, at $x = 0$, $\theta = \pi/2$, $b > \rho > a$:

$$V = 2\pi f(\rho). \tag{32}$$

In order to obtain (31) we have developed $[1 + (\rho^2/x^2)]^{-3/2}$ and $[1 + (x^2/\rho^2)]^{-3/2}$ by the binomial theorem. In integrating over the disc it has been assumed that we could still use these series when $\rho = x$. Although this procedure cannot be justified we shall see that we can get correct results by use of it.

We shall now consider a few special cases. As the simplest application of a double layer distribution is that of a magnetic shell we shall use the term "magnetic shell" in speaking of such distributions.

12. Consider first the case of a uniform circular magnetic shell. This is the simplest case whose solution gives us that of a uniform electric current flowing in a thin wire coinciding with the contour of the shell. The value of the potential, obtained by integrating (28) with $a = 0$, $f(\rho) = k$, expanding in a series and introducing zonal harmonics in the usual way, is:

$$V = 2\pi k \left\{ 1 - \frac{r}{b} P_1 + \frac{1}{2} \frac{r^3}{b^3} P_3 - \frac{1 \cdot 3}{2 \cdot 4} \frac{r^5}{b^5} P_5 + \cdots \right\} r < b. \tag{33}$$

If we now apply (31) to this case we get:

$$V = 2\pi k\left\{1 - \frac{r}{b}P_1 + \frac{1}{2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3}{2\cdot 4}\frac{r^5}{b^5}P_5 + \cdots\right\}$$
$$- 2\pi k\left\{1 - \left(1 + \frac{1}{2}\right)P_1 + \frac{3}{2}\left(\frac{1}{3} + \frac{1}{4}\right)P_3 - \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5} + \frac{1}{6}\right)P_5 + \cdots\right\}. \quad (34)$$

Hence we must have:

$$1 = P_0 = \left(1 + \frac{1}{2}\right)P_1 - \frac{3}{2}\left(\frac{1}{3} + \frac{1}{4}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5} + \frac{1}{6}\right)P_5 - \cdots. \quad (35)$$

The infinite series in (35) is divergent. Its divergency arises from the expansion of $[1 + (x^2/\rho^2)]^{-3/2}$ by the binomial theorem and making use of this expansion in integrating for $x = \rho$. We see now that this method leads to correct results if we define the value of the divergent series (35) as unity. For $\theta = 0$, $P_n = 1$, and the series is obviously equal to unity. For $\theta = \pi/2$, $P_1 = P_2 = P_3 = \cdots = 0$, and the numerical value of the series is 0. For any value of θ less than $\pi/2$ the series appears to approach more nearly to 1 the more terms that are used.

13. We consider next a circular ring-shaped magnetic shell whose strength is directly proportional to the radius. This case is easily realized practically. If we have fine wire wound in the form of a ring-shaped disc, of radii a and b, and send a unit electric current through it, the equivalent magnetic shell is a compound one consisting of the superposition of (1) a uniform magnetic shell of radius b, strength kb; (2) a uniform magnetic shell of radius a, strength $-ka$; (3) a variable ring-shaped magnetic shell, radii a and b, strength $-k\rho$. If N is the total number of turns of wire, $k = N/(b - a)$. The potential due to the first two distributions we can write down at once from the results already obtained. We now proceed to find the potential due to the third distribution.

The potential on the axis is:

$$V = 2\pi k\int_a^b \frac{x\rho^2 d\rho}{(x^2 + \rho^2)^{3/2}},$$

in which $k\rho$ is taken as the strength of the shell. This integral can be evaluated and expanded, giving:

$$V = 2\pi k\left\{x\log\frac{2b}{x} - x + b\left(\frac{1\cdot 3}{2\cdot 2}\frac{x^3}{b^3} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 4}\frac{x^5}{b^5} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 6}\frac{x^7}{b^7} - \cdots\right)\right.$$
$$\left. - a\left(\frac{1}{3}\frac{a^2}{x^2} - \frac{1\cdot 3}{2\cdot 5}\frac{a^4}{x^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{a^6}{x^6} - \cdots\right)\right\}.$$

Using equation (31) we find:

$$\begin{aligned} V = 2\pi k \Bigg\{ & rP_1 \log\frac{b}{r} + r\left(\frac{1}{3}P_1 - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right)P_5 - \cdots\right) \\ & + b\left(\frac{1\cdot 3}{2\cdot 2}\frac{r^3}{b^3}P_3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 6}\frac{r^7}{b^7}P_7 - \cdots\right) \\ & - a\left(\frac{1}{3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 3}{2\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right)\Bigg\}. \end{aligned}$$

If we write:

$$L_1 = \frac{1}{3}P_1 - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right)P_5 - \cdots \tag{36}$$

we see that

$$x \log\frac{2b}{x} - x$$

on the axis, expands into

$$rL_1 + rP_1 \log\frac{b}{r}.$$

Hence, putting $\theta = 0$, we find:

$$\log 2 = 1 + \frac{1}{3} - \frac{3}{2}\left(\frac{1}{2}+\frac{1}{5}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{4}+\frac{1}{7}\right) - \cdots. \tag{37}$$

Further, we find that L_1 is a solution of:

$$\frac{\partial^2 y}{\partial\theta^2} + \cot\theta\frac{\partial y}{\partial\theta} + 2y = 3P_1.$$

The solution of this equation, to agree with the infinite series (36) is:

$$L_1 = 1 - 2\cos\theta + \cos\theta\log\frac{4}{1+\cos\theta}. \tag{38}$$

The divergent series (36) which enters into the solution of this problem must be replaced by (38) in order to get correct results.

We can now write the value of the magnetic potential due to the distribution of electric currents described above. The result is:

$$\begin{aligned} V = 2\pi k \Bigg\{ & b\left(1 - \frac{r}{b}P_1 - \frac{1\cdot 1}{2\cdot 2}\frac{r^3}{b^3}P_3 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 4}\frac{r^5}{b^5}P_5 - \cdots\right) \\ & - a\left(\frac{1\cdot 1}{2\cdot 3}\frac{a^2}{r^2}P_1 - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{a^4}{r^4}P_3 + \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{a^6}{r^6}P_5 - \cdots\right) \\ & + r\left(P_1\log\frac{r(1+\cos\theta)}{4b} - 1 + 2P_1\right)\Bigg\} \quad a < r < b. \end{aligned}$$

We see that for $\theta = \pi/2$, $x = 0$, $r = \rho$:

$$V = 2\pi k(b - \rho)$$

showing that our solution satisfies (32).

14. As in the case of single layer distributions we can express the results, in a general form, of assuming the strength of the ring-shaped magnetic shell to be proportional to any integral positive or negative power of the radius.

If the strength of the shell varies directly as an even power or inversely as an odd power of the radius we find a divergent infinite series of zonal harmonics of odd order which must be taken as defining a zonal harmonic of even order:

$$\frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots(2n-1)}P_{2n} = \sum_{m=0}^{\infty}(-1)^{m+n}\frac{3\cdot 5\cdot 7\cdots(2m+1)}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n+1} + \frac{1}{2m+2n+2}\right)P_{2m+1}. \qquad (39)$$

For $n = 0$ we get (35) as a special case of this general formula.

15. If the strength of the magnetic shell varies directly as an odd power or inversely as an even power of the radius we get the following results by comparing the value of the potential obtained by integrating its value on the axis, and expanding, with that obtained by the use of (31).

$$\frac{3\cdot 5\cdot 7\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n}\log 2 = \frac{3\cdot 5\cdot 7\cdots 2n+1}{2\cdot 4\cdot 6\cdots 2n}\left(\frac{1}{2n+1} + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(2n-1)2n}\right) + \sum_{m=0}^{\infty}(-1)^{m+n}\frac{1\cdot 3\cdot 5\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n} + \frac{1}{2m+2n+3}\right). \qquad (40)$$

(37) is the special case of this for $n = 0$. Further, if we write the infinite series:

$$L_{2n+1} = \sum_{m=0}^{\infty}(-1)^{m+n}\frac{3\cdot 5\cdot 7\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m}\left(\frac{1}{2m-2n} + \frac{1}{2m+2n+3}\right)P_{2m+1} \qquad (41)$$

we have solutions of Laplace's equation in both the forms:

$$V = \frac{1}{r^{2n+2}}\left\{\frac{3\cdot5\cdot7\cdots2n+1}{2\cdot4\cdot6\cdots2n}P_{2n+1}\cdot\log r + L_{2n+1}\right\},$$

$$V = r^{2n+1}\left\{\frac{3\cdot5\cdot7\cdots2n+1}{2\cdot4\cdot6\cdots2n}P_{2n+1}\cdot\log r - L_{2n+1}\right\}.$$

L_{2n+1} satisfies the equation:

$$\frac{\partial^2 y}{\partial\theta^2} + \cot\theta\frac{\partial y}{\partial\theta} + (2n+1)(2n+2)y = (4n+3)\,\frac{3\cdot5\cdot7\cdots2n+1}{2\cdot4\cdot6\cdots2n}P_{2n+1}.$$

This is an equation of the same type as (21), whose solution, subject to the condition of remaining finite on the axis, we have found (23).

16. Another special case of interest is that of a magnetic shell whose strength is proportional to the logarithm of the radius. On the axis the potential is:

$$V = 2\pi kx\int_a^b\frac{\rho\log\rho d\rho}{(x^2+\rho^2)^{3/2}}.$$

Integrating, and developing, we find:

$$V = 2\pi k\left\{-\frac{a^2}{x^2}\left(\frac{\log a}{2}-\frac{1}{2^2}\right)+\frac{3a^4}{2x^4}\left(\frac{\log a}{4}-\frac{1}{4^2}\right)\right.$$
$$-\frac{3\cdot5}{2\cdot4}\frac{a^6}{x^6}\left(\frac{\log a}{6}-\frac{1}{6^2}\right)+\cdots-\frac{x}{b}(\log b+1)+\frac{3x^3}{2b^3}\left(\frac{\log b}{3}+\frac{1}{3^2}\right)$$
$$\left.-\frac{3\cdot5\cdot x^5}{2\cdot4b^5}\left(\frac{\log b}{5}+\frac{1}{5^2}\right)+\cdots+\log 2x\right\}.$$

When equation (31) is applied the result is:

$$V = 2\pi k\left\{-\frac{a^2}{r^2}\left(\frac{\log a}{2}-\frac{1}{2^2}\right)P_1+\frac{3a^4}{2r^4}\left(\frac{\log a}{4}-\frac{1}{4^2}\right)P_3\right.$$
$$-\frac{3\cdot5a^6}{2\cdot4r^6}\left(\frac{\log a}{6}-\frac{1}{6^2}\right)P_5+\cdots-\frac{r}{b}(\log b+1)P_1$$
$$+\frac{3r^3}{2b^3}\left(\frac{\log b}{3}+\frac{1}{3^2}\right)P_3-\frac{3\cdot5r^5}{2\cdot4b^5}\left(\frac{\log b}{5}+\frac{1}{5^2}\right)P_5+\cdots$$
$$+\log r\left[\left(1+\frac{1}{2}\right)P_1-\frac{3}{2}\left(\frac{1}{3}+\frac{1}{4}\right)P_3+\frac{3\cdot5}{2\cdot4}\left(\frac{1}{5}+\frac{1}{6}\right)P_5-\cdots\right]$$
$$\left.+\left(1-\frac{1}{2^2}\right)P_1-\frac{3}{2}\left(\frac{1}{3^2}-\frac{1}{4^2}\right)P_3+\frac{3\cdot5}{2\cdot4}\left(\frac{1}{5^2}-\frac{1}{6^2}\right)P_5-\cdots\right\}.$$

By (35) we see that the coefficient of $\log r$ is 1. If we write:

$$M_0 = \left(1 - \frac{1}{2^2}\right)P_1 - \frac{3}{2}\left(\frac{1}{3^2} - \frac{1}{4^2}\right)P_3 + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5^2} - \frac{1}{6^2}\right)P_5 - \cdots \tag{42}$$

we see that:

$$\log r + M_0$$

is a solution of Laplace's equation, and we find that M_0 satisfies the equation:

$$\frac{\partial^2 y}{\partial \theta^2} + \cot\theta \frac{\partial y}{\partial \theta} + 1 = 0.$$

The solution of this equation, finite on the axis, is:

$$M_0 = \log(1 + \cos\theta)$$

and it follows that:

$$\log 2 = \left(1 - \frac{1}{2^2}\right) - \frac{3}{2}\left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{5^2} - \frac{1}{6^2}\right) - \cdots, \tag{43}$$

an absolutely convergent series, as is (42).

17. The case just considered is of interest in connection with a distribution of electric currents flowing in concentric circles in a ring-shaped disc such that the current density at any point is inversely proportional to the radius of the point. Such a system of currents is produced in the Corbino effect, when a circular disc carrying a uniform radial current is placed in a normal magnetic field. The magnetic shell equivalent to this distribution is a compound one, consisting of (1) a uniform shell of radius a, strength $k \log b/a$, and (2) a variable ring-shaped shell of radii a and b, and strength $k \log b/\rho$. The potential due to such a compound shell in the region $a < r < b$ is:

$$V = 2\pi k \Bigg\{ \frac{r}{b}P_1 - \frac{1\cdot 1 r^3}{2\cdot 3 b^3}P_3 + \frac{1\cdot 1\cdot 3 r^5}{2\cdot 4\cdot 5 b^5}P_5 - \cdots$$
$$- \frac{1a^2}{2^2 r^2}P_1 + \frac{1\cdot 3a^4}{2\cdot 4^2 r^4}P_3 - \frac{1\cdot 3\cdot 5a^6}{2\cdot 4\cdot 6^2 r^6}P_5 + \cdots - \log\frac{r(1 + \cos\theta)}{b}\Bigg)\Bigg\}.$$

If C is the whole circular current, $C = k \log b/a$. When $\theta = \pi/2$, $x = 0$, we have

$$V = 2\pi k \log\frac{b}{\rho},$$

which agrees with (32).

18. We can now get the general results that follow from assuming that the strength of a ring-shaped magnetic shell is equal to

$$\frac{k \log \rho}{\rho^{2n+1}}.$$

If we define:

$$M_{2n} = \sum_{m=0}^{\infty} (-1)^{m+n+1} \frac{3\cdot 5\cdot 7\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m} \left\{ \frac{1}{(2n-2m-1)^2} - \frac{1}{(2n+2m+2)^2} \right\} P_{2m+1} \tag{44}$$

we find a solution of Laplace's equation:

$$V = \frac{1}{r^{2n+1}} \left\{ \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n} \log r + M_{2n} \right\}.$$

By assuming the strength of the shell to be $k\rho^{2n} \log \rho$, we find a corresponding solution:

$$V = r^{2n} \left\{ \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n} \log r - M_{2n} \right\}.$$

M_{2n} is a solution of the equation:

$$\frac{\partial^2 y}{\partial \theta^2} + \cot \theta \frac{\partial y}{\partial \theta} + 2n(2n+1)y = (4n+1) \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots 2n-1} P_{2n},$$

an equation which has already been solved. We find further:

$$\begin{aligned} \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1} \log 2 = {} & \frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots 2n-1} \\ & \times \left\{ \frac{1}{2n(2n-1)} + \frac{1}{(2n-2)(2n-3)} + \cdots + \frac{1}{2\cdot 1} \right\} \\ & - \sum_{m=0}^{\infty} (-1)^{m+n+1} \frac{3\cdot 5\cdot 7\cdots 2m+1}{2\cdot 4\cdot 6\cdots 2m} \\ & \times \left\{ \frac{1}{(2n-2m-1)^2} - \frac{1}{(2n+2m+2)^2} \right\}. \end{aligned} \tag{45}$$

The infinite series in this expression is absolutely convergent for all values of n, as is also the infinite series in (44). (43) is the particular case of (45) for $n = 0$. For $n = 1$ we find:

$$2 \log 2 = \frac{1}{16} + \frac{3}{2}\left(1 - \frac{1}{6^2}\right) - \frac{3\cdot 5}{2\cdot 4}\left(\frac{1}{3^2} - \frac{1}{8^2}\right) + \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\left(\frac{1}{5^2} - \frac{1}{10^2}\right) - \cdots.$$

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